Mass inversion in a critical neutron star: An alternative to the black hole model.

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SECOND PART

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Abstract : In this second part, we describe the second solution to the Einstein field equations published by Karl Schwarzschild in February 1916, which refers to a geometry within a sphere of constant density, a paper that has been translated in English only in Late 1999, still ignored by most black hole specialists. It is shown that Schwarzschild perfectly identified and described the outbreak of a physical criticality occurring in the center of a massive star a century ago, before the geometric criticality comes into play. On this basis, a new model has been built, alternative to the stellar black hole model, describing the behavior of a subcritical neutron star destabilized by an excess of matter incoming from a companion star. The "frozen time" issue, classically associated with the phenomenon, is dealt by taking into account frame-dragging effects due to the rotation of the object in the Kerr metric.

Introduction

In the first part of this article, we have re-examined the solution to the Einstein field equations found by Karl Schwarzschild in 1916 [1] from a mathematical angle, showing that its analytic continuations are nothing but extensions into a purely imaginary realm.

If we admit this idea belongs to Physics, then we must take into account all that has been concocted for half a century with respect to this "interior of black holes", in particular their "thermodynamics" and the "central singularity". As for the alteration of the metric signature going from $(+---)$ to $(-+--)$ when the surface of the event horizon is crossed, it is recalled this is conventionally interpreted by saying that inside the black hole, t becomes a space variable, and r a time.

Before considering a *real* extension of this solution, we shall consider the second paper published by Schwarzschild in February 1916. [2]

424 Sitzung der phys.-math. Klasse v. 23. März 1916. — Mitt. v. 24. Februar

Über das Gravitationsfeld einer Kugel aus inkompressibler Flüssigkeit nach der Einsteinschen Theorie.

Von K. Schwarzschild.

(Vorgelegt am 24. Februar 1916 [s. oben S. 313].)

§ 1. Als ein weiteres Beispiel zur Einsteinschen Gravitationstheorie

Fig. 1 – Karl Schwarzschild's second paper, 24 February 1916: "**On the gravitational field of a sphere of incompressible fluid according to Einstein's theory.**"

Since this essential article was not available in English until December 1999, [2] it is quite likely that black hole specialists are unaware of its content, and perhaps even its existence.

Der »kovariante Energietensor« wird:

$$T_{\mu\nu} = \sum_{\tau} T_{\mu}^{\tau} g_{\nu\tau} \,. \qquad (3)$$

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Es sei noch:

$$T = \sum_{\sigma} T_{\sigma}^{\sigma} = \rho_0 - 3p \qquad (4)$$

und:

$$\varkappa = 8\pi k^2,$$

wo k^2 die Gausssche Gravitationskonstante ist. Dann lauten nach Hrn. Einstein (diese Berichte 1915, S. 845, Gl. 2a) die rechten Seiten der Feldgleichungen:

$$G_{\mu\nu} = -\varkappa\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right). \qquad (5)$$

Fig 2 –Schwarzschild takes Einstein's equations over.

In equation (3) we recognize the covariant form of the energy-momentum tensor. In (4) its trace. And just below, Einstein's constant, expressed using a speed of light taken equal to unity. In (5) the field equation Einstein has just published a few months earlier.

$$x_1 = \frac{r^3}{2}, \quad x_2 = -\cos\vartheta, \quad x_3 = \phi, \quad x_4 = t \qquad (7)$$

einzuführen. Das Linienelement muß dann, wie dort, die Form haben:

$$ds^2 = f_4 dx_4^2 - f_1 dx_1^2 - f_2 \frac{dx_2^2}{1 - x_2^2} - f_2 dx_3^2 (1 - x_2^2), \qquad (8)$$

so daß man hat:

$$g_{11} = -f_1, \quad g_{22} = -\frac{f_2}{1 - x_2^2}, \quad g_{33} = -f_2(1 - x_2^2), \quad g_{44} = f_4$$

$$(\text{die übrigen } g_{\mu\nu} = 0).$$

Dabei sind die f Funktionen nur von x_1.

Auch ergeben sich für den Raum außerhalb der Kugel die dortigen Lösungen (10), (11), (12):

$$f_4 = 1 - \alpha(3x_1 + \rho)^{-1/3}, \quad f_2 = (3x_1 + \rho)^{2/3}, \quad f_1 f_2^2 f_4 = 1, \qquad (9)$$

wobei α und ρ zwei zunächst willkürliche Konstanten sind, die sich weiter-

Fig. 3

With $r = \sqrt{x^2 + y^2 + z^2}$, after renewing Einstein's hypothesis about the choice of the determinant of the metric $g_{11}\, g_{22}\, g_{33}\, g_{44} = -1$ he takes again almost in the same form except an additional coefficient in (7) his choice of coordinates already implemented the month before in his exterior solution (equation (7) of the "Massenpunkt" paper [1]). In doing so, he has positive functions f_1, f_2, f_3, f_4 which gives a metric signature $(+---)$ like his exterior solution. Equations (9) are identical to the choice made in equation (10) of his previous paper. He specifies that α and ρ are arbitrary constants that will have to be determined later, using the mass and the radius of the star

Equation (8) made explicit:

$$ds^2 = f_4\, dt^2 - \frac{3}{2} f_1\, dr^2 - f_2\, (\, d\vartheta^2 + \sin^2\vartheta\, d\phi^2 \,)$$

which a a form enabling to rejoin easily the exterior solution to the surface of the star.

At the end of page 428, he writes:

Somit ist unser Problem auf Quadraturen zurückgeführt.

§ 5. Es sind nun die Integrationskonstanten so zu bestimmen, daß das Innere der Kugel singularitätenfrei bleibt und an der Kugeloberfläche der stetige Anschluß an die Außenwerte der Funktionen f und ihrer Derivierten bewirkt wird.

Fig. 4

Translation:

§5. The integration constants must now be determined in such a way that the interior of the sphere remains free from singularities and the continuous junction to the external values of the functions f and of their derivatives at the surface of the sphere is realised.

in order to insure the continuity of geodesics.

Making the auxiliary quantity R appear according to $r = (R^3 + \alpha^3)^{1/3}$ he notices that outside the sphere, the metric represents the point mass, cf. his January 1916 *"Massenpunkt"* paper. [1]

Außerhalb der Kugel bleibt die Form des Linienelements dieselbe, wie beim Massenpunkt:

$$\left.\begin{aligned} ds^2 &= \left(1 - \frac{\alpha}{R}\right) dt^2 - \frac{dR^2}{1 - \alpha/R} - R^2 (d\vartheta^2 + \sin^2\vartheta\, d\phi^2) \\ \text{wobei:}\quad & R^3 = r^3 + \rho \end{aligned}\right\} \quad (36)$$

ist. Nur wird ρ nach (33) bestimmt, während für den Massenpunkt $\rho = \alpha^3$ war.

Fig. 5 – "Outside the sphere the form of the line element remains the same as in Mass point".

In the rest of his calculation, Schwarzschild details all parameters related to the solution. His study is very complete.

He writes:

2. Man entnimmt den Bewegungsgleichungen eines Punktes von unendlich kleiner Masse außerhalb unserer Kugel, welche dieselbe Form wie beim Massenpunkt (dortige Gleichungen (15)—(17)) behalten, folgende Bemerkungen:

In großer Entfernung erfolgt die Bewegung des Punktes nach dem Newtonschen Gesetz, wobei $\alpha/2k^2$ die Rolle der anziehenden Masse spielt. Es kann daher $\alpha/2k^2$ als »Gravitationsmasse« unserer Kugel bezeichnet werden.

Läßt man ferner einen Punkt aus der Ruhe im Unendlichen bis zur Kugeloberfläche herabfallen, so erhält die »natürlich gemessene« Fallgeschwindigkeit den Betrag:

$$v_a = \frac{1}{\sqrt{1-\alpha/R}} \frac{dR}{ds} = \sqrt{\frac{\alpha}{R_a}}.$$

Es ist also nach (40):

$$v_a = \sin \chi_a. \tag{42}$$

Für die Sonne ist die Fallgeschwindigkeit rund $^1/_{500}$ Lichtgeschwindigkeit. Man überzeugt sich leicht, daß bei dem kleinen hieraus sich ergebenden Wert von χ_a und χ ($<\chi_a$) alle unsre Gleichungen bis auf die bekannten Einsteinschen Effekte zweiter Ordnung in die der Newtonschen Theorie übergehen.

3. Für das Verhältnis der Gravitationsmasse $\alpha/2k^2$ zur substan-

Fig. 6

Translation:

2. About the equations of motion of a point of infinitely small mass outside our sphere, which maintain the same form as in “Mass point” (there equations (15)-(17)), one makes the following remarks:

For large distances the motion of the point occurs according to Newton’s law, with $\alpha/2k^2$ playing the rôle of the attracting mass. Therefore $\alpha/2k^2$ can be designated as “gravitational mass” of our sphere.

If one lets a point fall from the rest at infinity down to the surface of the sphere, the “naturally measured” fall velocity takes the value:

$$v_a = \frac{1}{\sqrt{1-\frac{\alpha}{R}}} \frac{dR}{ds} = \sqrt{\frac{\alpha}{R^a}}$$

Hence, due to (40):

$$v_a = \sin \chi_a$$

For the Sun the fall velocity is about 1/500 the velocity of light. One easily satisfies himself that, with the small value thus resulting for χ_a and χ ($< \chi_a$), all our equations coincide with the equations of Newton's theory apart from the known second order Einstein's effects.

The end of his paper has our undivided attention. The angle χ allows to locate oneself inside the sphere. $\chi = 0$ is the geometric center. He writes:

4. The velocity of light in our sphere is

$$v = \frac{2}{\cos \chi_a - \cos \chi}$$

hence it grows from the value $1/\cos \chi_a$ at the surface to the value $2/(3\cos \chi_a - 1)$ at the center. The value of the pressure quantity $\varrho_0 + p$ according to (10) and (30) grows in direct proportion to the velocity of light.

Moreover:

durch Ausstrahlung) erfolgt.

4. Die Lichtgeschwindigkeit in unserer Kugel wird:

$$v = \frac{2}{3\cos \chi_a - \cos \chi}, \qquad (44)$$

sie wächst also vom Betrag $\frac{1}{\cos \chi_a}$ an der Oberfläche bis zum Betrag $\frac{2}{3\cos \chi_a - 1}$ im Mittelpunkt. Die Druckgröße $\rho_0 + p$ wächst nach (10) und (30) proportional der Lichtgeschwindigkeit.

Im Kugelmittelpunkt ($\chi = 0$) werden Lichtgeschwindigkeit und Druck unendlich, sobald $\cos \chi_a = 1/3$, die Fallgeschwindigkeit gleich $\sqrt{8/9}$ der (natürlich gemessenen) Lichtgeschwindigkeit geworden ist. Es

ist damit eine Grenze der Konzentration gegeben, über die hinaus eine Kugel inkompressibler Flüssigkeit nicht existieren kann. Wollte man unsere Gleichungen auf Werte $\cos \chi_a < 1/3$ anwenden, so erhielte man bereits außerhalb des Kugelmittelpunktes Unstetigkeiten. Man kann jedoch für größeres χ_a Lösungen des Problems finden, welche wenig-

Fig. 7

i.e.:

> At the center of the sphere ($\chi = 0$) velocity of light and pressure become infinite when $\cos \chi_a = 1/3$, and the fall velocity becomes 8/9 of the (naturally measured) velocity of light. Hence there is a limit to the concentration, above which a sphere of incompressible fluid can not exist. If one would apply our equations to values $\cos\chi a < 1/3$, one would get discontinuities already outside the center of the sphere.

Finally, note the last sentence of the paper:

und Bewegung innerhalb unsres Gravitationsfeldes keine Rolle spielt. Für einen außen messenden Beobachter folgt gemäß (40), daß eine Kugel von gegebener Gravitationsmasse $\alpha/2k^2$ keinen kleineren außen gemessenen Radius haben kann, als:

$$P_a = \alpha .$$

Für eine Kugel aus inkompressibler Flüssigkeit wird die Grenze $9/8\,\alpha$. (Für die Sonne wird α gleich 3 km, für eine Masse von 1 g gleich $1.5 \cdot 10^{-28}$ cm.)

Fig. 8

i.e.:

> For an observer measuring from outside it follows from (40) that a sphere of given gravitational mass $\alpha/2k^2$ can not have a radius measured from outside smaller than:
>
> $$P_o = \alpha$$
>
> For a sphere of incompressible fluid the limit will be $9/8\ \alpha$. (For the Sun α is equal to 3 km, for a mass of 1 gram is equal to $1.5 \cdot 10^{-28}$ cm.)

A physical criticality before the geometric criticality

Thus, as early as February 1916, Karl Schwarzschild had detected that a peculiar situation of *physical criticality* (where the pressure and speed of light become infinite in the center of the star) manifests itself *before* the classical *geometric criticality* is reached (when the radius of the star merges with the Schwarzschild radius).

From his article, it is then easy to calculate the value of the pressure as a function of the outer radius of the star.

This corresponds to the famous equation established in 1939, based on Richard Tolman's work [3], by J. Robert Oppenheimer and George Volkoff [4] which is the basis of the "TOV model" named after their initials:

$$\frac{dp}{dr} = -\frac{(\rho + p)(m(r) + 4\pi\, k\, p\, r^3)}{r\,(r - m(r))}$$

This TOV model considers, like Schwarzschild's model from which it is derived, a neutron star as a sphere of constant density.

Fig. 9 is a line graph displaying solution curves of the TOV equation. It shows the evolution of pressure in the neutron star according to a logarithmic scale and as a function of the distance from the center of the star, for different values of its outer radius R_n (hence its mass):

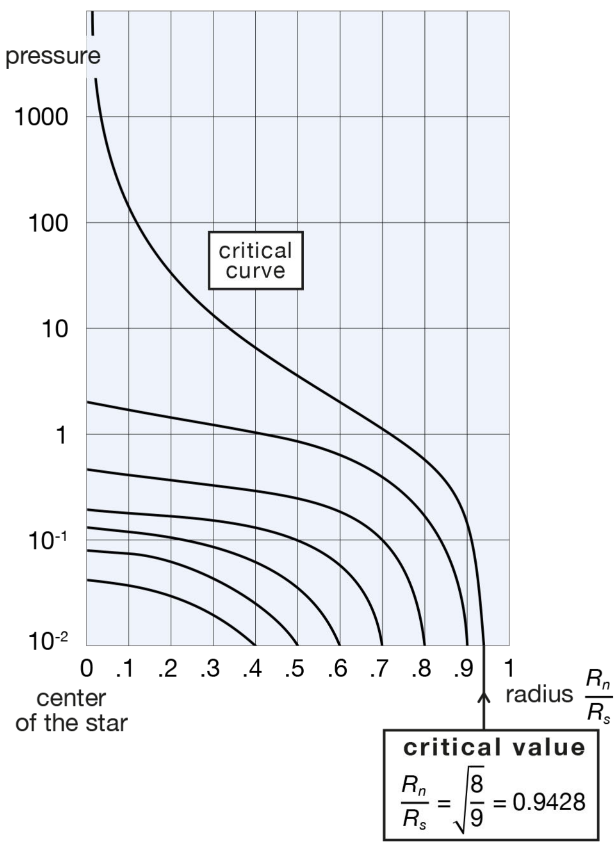

Fig. 9 – Pressure inside a neutron star
as a function of its radius (TOV model).

It can be seen that the pressure at the center of the star tends towards infinity when $R_n = 0.9428\, R_s$ thus before the radius of the star reaches the Schwarzschild radius.

The solution for values greater than this critical radius is shown in Figure 10. Although the TOV solution is steady-state and assumes a spherical symmetry, thus represents only an approximation, it denotes the extreme fast growth rate of the singularity (p = ∞) that comes into being at the center of the star, the more the Schwarzschild radius is approached:

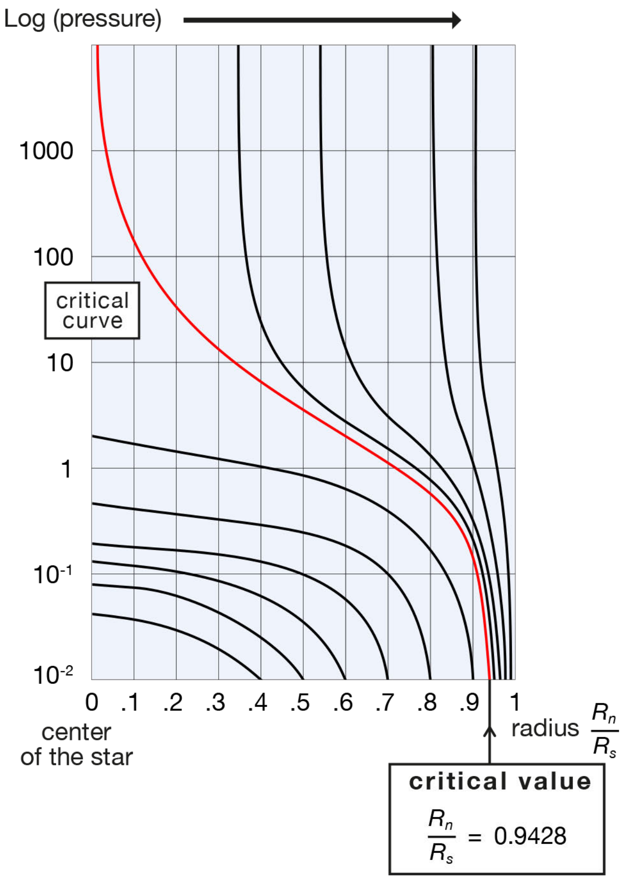

Fig. 10 – Pressure soaring quickly in the neutron star for values greater than the critical value.

Those who build black hole theoretical models start from the hypothesis in which the star would *shrink asymptotically down to the Schwarzschild radius*.

The following curve shows how this situation is achieved. Along abscissa is the radius of the star R_n. The parabola represents the evolution of the Schwarzschild radius R_s which increases according to the mass of the star, hence, for a constant density, as the radius of the star cubed.

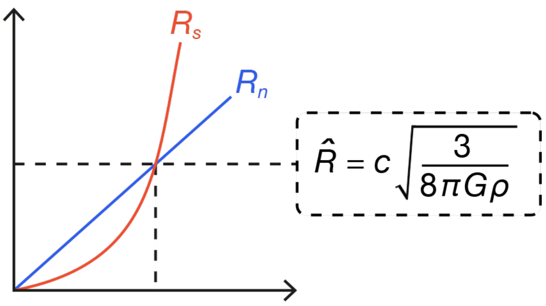

Fig. 11 – Rapid growth of the Schwarzschild radius R_s
with respect to the linear growth of the radius of the star R_n.
$\hat{R}$ is the critical star radius when $R_n = R_s$ (geometric criticality).

We observe a horizontal line, directly derived from the Schwarzschild solution of February 1916, which marks the limit of criticality of the inner metric.

Below this critical situation, the neutron star is like this:

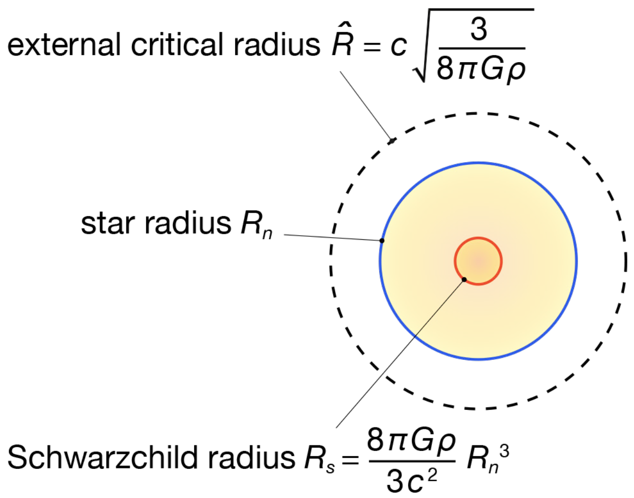

Fig. 12

We see that its Schwarzschild radius R_s is located inside the star, whereas the radius $\hat{R}$, which marks the criticality of the external solution, lies outside. Thus the two solutions, connected outside this geometric criticality, are singularity-free.

Theorists completely obscure the physical criticality, i.e. the tremendous rises of pressure and speed of light to infinity in the center of the star, which take place before the situation of geometric criticality occurs. This should not leave the physicist unmoved, as a pressure is also an energy density.

Reconsideration of the problem according to the Janus model

The Schwarzschild exterior and interior solutions are also solutions of the Janus cosmological model, [5] [6] [7] a bimetric description of the universe as an M_4 manifold associated to two conjugated Riemannian metrics, generating their own set of geodesics, solutions of two coupled field equations:

$$R_{\mu\nu}^{(+)} - \frac{1}{2} R^{(+)} g_{\mu\nu}^{(+)} = \chi \left[T_{\mu\nu}^{(+)} + \sqrt{\frac{g^{(-)}}{g^{(+)}}} T_{\mu\nu}^{(-)} \right]$$

$$R_{\mu\nu}^{(-)} - \frac{1}{2} R^{(-)} g_{\mu\nu}^{(-)} = -\chi \left[\sqrt{\frac{g^{(+)}}{g^{(-)}}} T_{\mu\nu}^{(+)} + T_{\mu\nu}^{(-)} \right]$$

Let's start with the Schwarzschild exterior solution, with no right hand-side. It is known that in the Schwarzschild solution, the quantity α is just a simple integration constant, which can be taken positive or negative.

In the following, a local situation is arbitrarily considered, where $g^{(+)} = g^{(-)}$. Let us assume that the length R_s, the Schwarzschild radius, is strictly positive and that α, simple integration constant, can have equal and opposite values. Thus we have two coupled solutions outside the star. Let's write these metrics:

$$g_{\mu\nu}^{(+)ext} \qquad g_{\mu\nu}^{(-)ext}$$

Let's call them "posi-Schwarzschild exterior" and "nega-Schwarzschild exterior". They are written, choosing Schwarzschild's auxiliary quantity R or "Hilbert's variable":

Metric $g_{\mu\nu}^{(+)ext}$	$ds^2 = (1 - \frac{R_s}{R})\, dt^2 - \frac{dR^2}{1 - \frac{R_s}{R}} - R^2 (d\theta^2 + \sin^2\theta\, d\varphi^2)$
Metric $g_{\mu\nu}^{(-)ext}$	$ds^2 = (1 + \frac{R_s}{R})\, dt^2 - \frac{dR^2}{1 + \frac{R_s}{R}} - R^2 (d\theta^2 + \sin^2\theta\, d\varphi^2)$

We can easily calculate the geodesics of the second equation and see that they bring up the repulsion of a test mass +*m* by a negative mass -*m*.

It is the same thing for "posi-Schwarzschild interior" $g_{\mu\nu}^{(+)\mathrm{int}}$ and "nega-Schwarzschild interior" $g_{\mu\nu}^{(-)\mathrm{int}}$ which are written, [1] derived from the Schwarzschild interior solution, February 1916:

Metric $g_{\mu\nu}^{(+)\mathrm{int}}$
$$ds^2 = \left[\frac{3}{2}\sqrt{1-\frac{R_n^2}{\hat{R}^2}} - \frac{1}{2}\sqrt{1-\frac{R^2}{\hat{R}^2}} \right] dt^2 - \frac{dR^2}{1-\dfrac{R^2}{R_s^2}} - R^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Metric $g_{\mu\nu}^{(-)\mathrm{int}}$
$$ds^2 = \left[\frac{3}{2}\sqrt{1+\frac{R_n^2}{\hat{R}^2}} - \frac{1}{2}\sqrt{1+\frac{R^2}{\hat{R}^2}} \right] dt^2 - \frac{dR^2}{1+\dfrac{R^2}{R_s^2}} - R^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

These metrics are connected in pair along the surface of the star.

What happens, according to the Janus model, when the mass of a subcritical neutron star is gradually increased to the point where the physical criticality is reached at its center? As we will see in the next section, this kind of event should be quite common in the cosmos.

It must be said that without a complete time-dependent analytic solution that could be based on a previous work, we will give this article a somewhat conjectural character.

But in this day and age of evaporating black holes fitted with firewalls protecting them from the information paradox, we think we can grant ourselves this right.

Leaking Neutron Star

First and foremost, we present the "soft" scenario, where such a physical criticality is slowly approached.

As already mentioned in the first part of this paper, binary systems are very abundant in the universe. For a number of them, a member of the couple has become a neutron star after the gravitational collapse of a supernova.

[1] See reference [8] and more precisely [9], equation (14.47) page 472, chapter 14 *"The Role of Relativity in Stellar Structure and Gravitational Collapse*".

In this particular case, this neutron star is stable, subcritical. Then its donor companion star sends a continuous flow of matter in the form of stellar wind, accreted gravitationally by the neutron star, whose mass is therefore gradually increased.

The rise of the energy density at the center of the star eventually produces there a mass inversion process. Indeed, according to the Janus cosmological model, all physical constants underwent a joint variation during the high energy density state of the radiation-dominated era, right after the Big Bang where *c*, for example, reached an infinite value. [6] We think some events involving very high energy densities, as in the center of critical neutron stars, can similarly recreate conditions allowing a "bridge" to briefly appear, joining the positive and negative sectors together, though which mass can be exchanged and inverted. [11]

As this mass in excess, which has become negative according to the Janus model, no longer interacts with the positive mass except through antigravitation, it is repelled by the star and dispersed away in space among the interstellar medium, then the intergalactic medium where it ends up.

An analogy can be made, comparing this process with the "bung & float" mechanism keeping the water level constant in a flush tank:

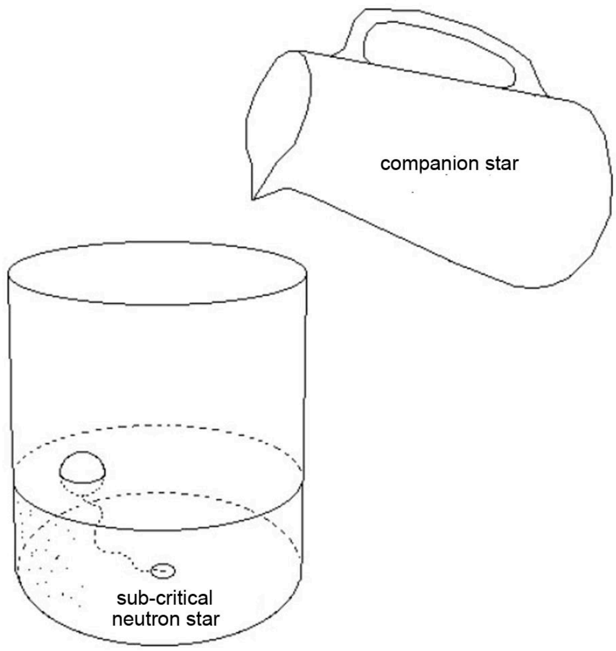

Fig. 13

When an input of matter in excess triggers the beginning of a criticality in the center of the star, "the bung is raised" and the "plughole" opens:

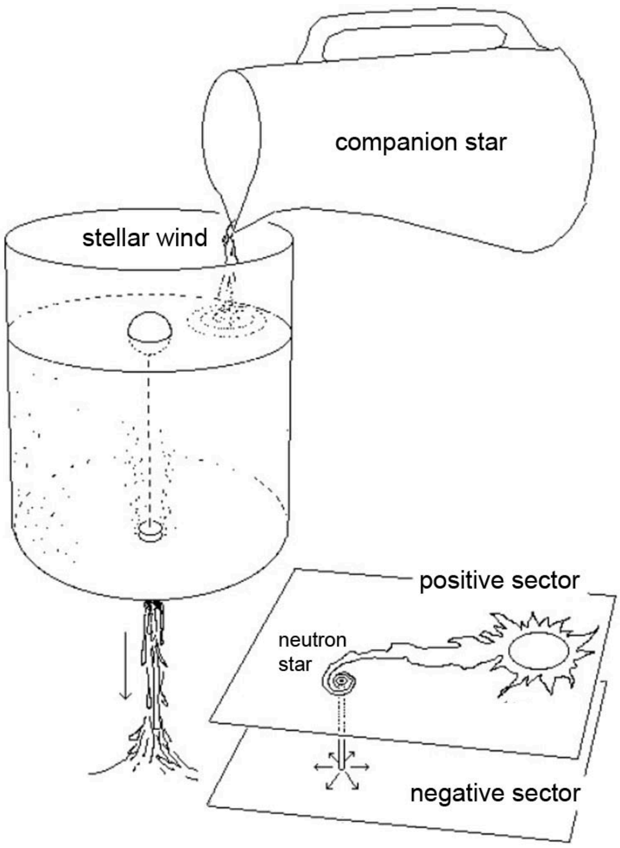

Fig. 14

The plughole, quite small in diameter, quickly closes as soon as the star has become subcritical again, and the situation becomes:

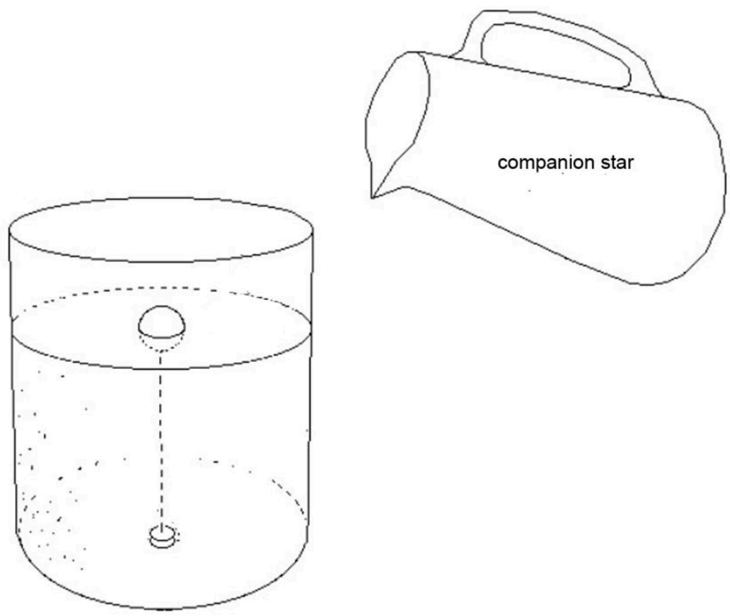

Fig. 15

When a model is proposed, is must describe a possible observation able to falsify it. In this case, it is a non-observation: we conjecture that stellar black holes do not exist and that the observed effects are only caused by subcritical neutron stars, which are abundant.

The geometry describing these objects is then represented by the four metrics given above, a borderline situation at the edge of criticality, which corresponds, in Schwarzschild's second paper, to:

$$\cos \chi_a \text{ slightly superior to } 1/3$$

We can then give a name to such kind of neutron stars near criticality, smoothly evacuating any input of matter in excess by the inversion of some of its mass at its center:

Leaking neutron star

A more faithful image would be to shape the flush tank as a bowl, corresponding to the gravitational potential in the neutron star. As long as the criticality is not reached, we are in the situation of Fig. 16 on the left. When the critical mass is reached, the bung is raised. The matter in excess, whose mass is reversed, is poured out. It follows the relief of a promontory which is the mirror image of the bowl (the gravitational potential is reversed after its passage in this sector). This, until the level drops under criticality, causing the bung to close.

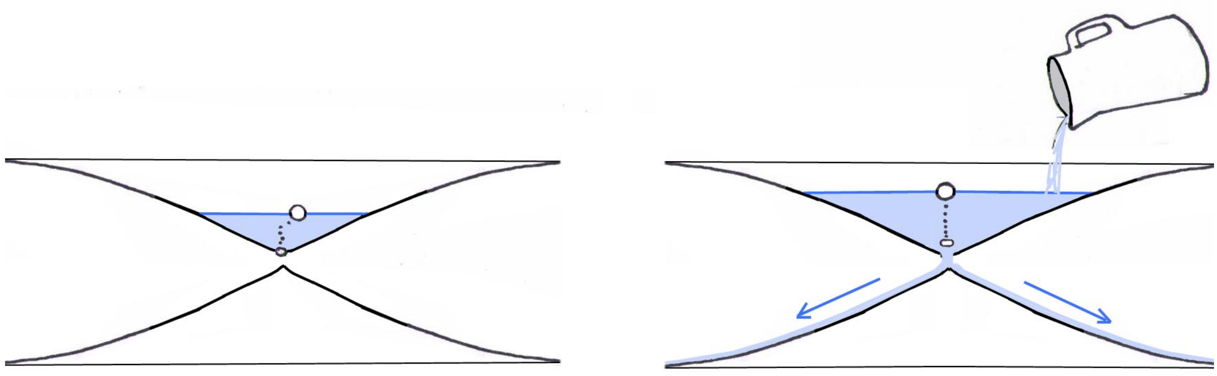

Fig. 16

Interior coupled metrics are then:

Metric $g_{\mu\nu}^{(+)\text{int}}$ $$ds^2 = \left[\frac{3}{2}\sqrt{1-\frac{9R_s^2}{8\hat{R}^2}} - \frac{1}{2}\sqrt{1-\frac{R^2}{\hat{R}^2}} \right] dt^2 - \frac{dR^2}{1-\frac{R^2}{R_s^2}} - R^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Metric $g_{\mu\nu}^{(-)\text{int}}$ $$ds^2 = \left[\frac{3}{2}\sqrt{1+\frac{9R_s^2}{8\hat{R}^2}} - \frac{1}{2}\sqrt{1+\frac{R^2}{\hat{R}^2}} \right] dt^2 - \frac{dR^2}{1+\frac{R^2}{R_s^2}} - R^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Formulation unchanged for exterior metrics:

Metric $g_{\mu\nu}^{(+)ext}$ $$ds^2 = (1-\frac{R_s}{R})\, dt^2 - \frac{dR^2}{1-\frac{R_s}{R}} - R^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Metric $g_{\mu\nu}^{(-)ext}$ $$ds^2 = (1+\frac{R_s}{R})\, dt^2 - \frac{dR^2}{1+\frac{R_s}{R}} - R^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Collapse of neutron binaries

A "hard" scenario corresponds to the fusion of two subcritical neutron stars into a single object. Let us call M_1 and M_2 the masses of these two stars. If $M_1 + M_2 < 2.5$ solar masses, this fusion will take place without mass inversion. But if the sum exceeds this value, the mass in excess m will be inverted and repelled from the resulting star, which will become subcritical again. Let us call this sum: $M_1 + M_2 + m$.

The inversion and expulsion of this excess of mass m, brief and brutal, must be accompanied by a powerful emission of gravitational waves. This is our interpretation of the recent evidence at LIGO [10] and not as a fusion of two black holes about thirty solar masses. Values producing a gravitational signal with an energy equivalent to only 3 solar masses allow such an identification in spite of a tiny signal-to-noise ratio.

We will then seek to describe this process building a steady state solution. Already, the high sensitivity of the solution when approaching criticality evokes the rapid expansion of a central devouring singularity, reminiscent of the ancient video game Pac-Man, then its closing just as fast.

An extreme situation would be encountered with the fusion of two subcritical neutron stars. The total inverted mass then approaches 2.5 solar masses. Once the mass inversion process has been completed, how could such a situation be described, geometrically?

This situation is not simple. In this type of scenario, and in certain cases, a configuration can be considered with the system of equations

$$R_{\mu\nu}^{(+)} = 0$$

$$R_{\mu\nu}^{(-)} = 0$$

which would represent a kind of "instantaneous photography" where the mass density is not zero locally, but the sum of the two mass-energy densities, positive and negative, is equal to zero. In reference [11] this type of solution was studied, through joint steady state metric solutions. Although this is not perfectly rigorous yet, it can still give a general idea. This is an open and exciting question that we are currently working on.

Such a solution was described in 2015. [11] When the system of the four metric solutions of the Janus model is considered, it should be noted that one can opt for a solution where the transferred mass, which has become negative, is now located in the negative sector.

This configuration would then be:

Metric $g_{\mu\nu}^{(+)\mathrm{int}}$	$ds^2 = \left[\frac{3}{2}\sqrt{1+\frac{R_s^2}{\hat{R}^2}} - \frac{1}{2}\sqrt{1+\frac{R_s^2}{R^2}} \right] dt^2 - \frac{dR^2}{1+\frac{R_s^2}{R^2}} - R^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$
Metric $g_{\mu\nu}^{(-)\mathrm{int}}$	$ds^2 = \left[\frac{3}{2}\sqrt{1-\frac{R_s^2}{\hat{R}^2}} - \frac{1}{2}\sqrt{1-\frac{R_s^2}{R^2}} \right] dt^2 - \frac{dR^2}{1-\frac{R_s^2}{R^2}} - R^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$
Metric $g_{\mu\nu}^{(+)ext}$	$ds^2 = (1+\frac{R_s}{R})\, dt^2 - \frac{dR^2}{1+\frac{R_s}{R}} - R^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$
Metric $g_{\mu\nu}^{(-)ext}$	$ds^2 = (1-\frac{R_s}{R})\, dt^2 - \frac{dR^2}{1-\frac{R_s}{R}} - R^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$

Compared to the configuration where the mass is positive, there is a complete exchange of the two geodesic systems. By the way it can be noted that this system of four metrics describes the geometry in the vicinity of negative mass conglomerates located at the center of the giant voids of the large-scale structure of the universe, as foreseen by the Janus model. This configuration generates negative gravitational lensing on positive energy photons, a phenomenon initially described by the author in 1995. [12] In this model, a gravitational lens effect reduces the apparent magnitude of high-redshift galaxies ($z > 7$) which makes them *appear* as dwarfs.

As suggested by a Japanese team, [13] and in the video "JANUS #20" presenting these concepts, [14] mapping the universe using *negative weak gravitational lensing* could falsify this model of lacunar large-scale structure of the universe, with galaxy filaments, clusters and superclusters around large void bubbles. [15]

But this neutron star model remains, as it is, schematic and embryonic. It is known that such objects are rotating fast, which is not currently taken into account. The external metric should therefore be that of Kerr, [16] and not that of Schwarzschild, and the corresponding internal metric remains to be made.

Moreover, the magnetic field associated with neutron stars is very intense: typically 10^8 teslas, to more than 10^{11} teslas for most extreme cases (magnetars). It results from the compression of magnetic field lines that preexisted in the massive star, before its gravitational collapse in the supernova phenomenon. This magnetic field probably plays an important role, likely to influence the proposed scenario, in particular on the structure of possible negative mass remnants resulting from the violent fusion and mass inversion of a pair of neutron stars.

Nevertheless, a study carried out on the basis of the Schwarzschild metric enables the identification of general ideas.

A change of topology

This approach modifies the idea commonly followed in the field of differential geometry and manifold theory. Remember that these theories were born from concerns about how to map Earth, which is a sphere. However, in order to map it, only plane paper sheets are available. Cartographers therefore use atlases made up of flat maps, supplied with indications allowing to connect them two by two.

Modern geometry has attempted to generalize this idea. But imagine a planet where planar surfaces, in particular paper as a map support, are unknown. Conversely, there is in this world a profusion of tree leaves shaped as spherical caps, stackable one onto the others. Cartographers of this planet could then constitute an atlas made of sets of maps traced onto curved sheets.

Differential geometry has made widespread use of the set "atlas + maps" which allows to shape those maps onto any support, as long as rules are provided to connect them to each other.

In such a perspective, the approach represents:

- The choice of a manifold M_n.
- Its arbitrary association to a topology.
- The addition of maps constituting an atlas.

When he builds his solution, Schwarzschild opts implicitly for a representation space $\mathbb{R}^3 \times \mathbb{R}$. He intends to describe his hypersurface-solution in such a space. He then obtains a metric that evokes a non-contractible object, that may be taken for a manifold with boundary.

Hilbert, giving to time the nature of a pure imaginary quantity,[2] implicitly situates the solution in a space $\mathbb{R}^3 \times \mathbb{C}$.

The analytic extensions developed later, like that of Kruskal, [17] belong to the same choice, to the same idea. This is what will make the Argentinian-American physicist Juan Martín Maldacena say:[3]

— *Kruskal extended the solution to cover the full spacetime.*

Specifically, he should have said:

— *Kruskal extended the solution to cover his views of the spacetime.*

In doing so, he builds this extension using several connected metrics.

[2] Please refer to " Mass inversion in a critical neutron star: An alternative to the black hole model – First part".

[3] https://indico.fias.uni-frankfurt.de/event/4/session/17/contribution/39

An important question arises:

— What is the *information* contained in a solution to the field equation, expressed in the form of a metric? Would such a solution contain *its own topology*?

Said otherwise:

— Should the expression of this solution, in the form of a hypersurface, be limited to representations where the element of length remains real?

The Einstein field equations, whose metric is derived, involves only real quantities. The components of the Ricci tensor are real, its derivatives are real. How, under these conditions, to consider expressing the solution by implementing pure imaginary quantities?

If we stick to the *real* world, this means that when we explore parts of the (numerical) space related to variables, and come across an element of pure imaginary length, or when the signature of the metric becomes altered, which is the same thing, one is then simply *outside the hypersurface*.

This can be illustrated with the Schwarzschild solution. It is then enough to start from "Hilbert's representation" with the auxiliary variable *R*:

$$ds^2 = \left(1 - \frac{R_s}{R}\right) dt^2 - \frac{dR^2}{1 - \frac{R_s}{R}} - R^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right)$$

and use the change of space variable as shown in ref. [11]:

$$R = R_s \left(1 + Log\, ch\, \rho \right)$$

The metric is then written:

$$ds^2 = \frac{Log\, ch\, \rho}{1 + Log\, ch\, \rho} dt^2 - R_s^2 \left[\frac{1 + Log\, ch\, \rho}{Log\, ch\, \rho} th^2 \rho \, d\rho^2 + (1 + Log\, ch\, \rho)^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) \right]$$

This metric is regular for any value of this new space variable ρ including the vicinity of $\rho = 0$ equivalent to $R = R_s$ with $Log\, ch\, \rho = 0$.

To get convinced, just make series expansion of functions $Log\, ch\, \rho$ and $th^2 \rho$ in the neighborhood of $\rho = 0$ to find their ratio tends to 2.

What is the topology of this four-dimensional hypersurface? We can focus on its spatial part:

$$d\sigma^2 = R_s^2 \left[\frac{1+Log\,ch\,\rho}{Log\,ch\,\rho} th^2\rho\,d\rho^2 + (1+Log\,ch\,\rho)^2 (d\theta^2 + \sin^2\theta\,d\varphi^2) \right]$$

This geometric object is not contractible. If we fix the quantity ρ we obtain an object defined by the metric:

$$d\Sigma^2 = R_s^2 (1+Log\,ch\,\rho)^2 (d\theta^2 + \sin^2\theta\,d\varphi^2)$$

By setting the variable θ, for example to the value $\theta = \pi/2$, we get a maximum value for the perimeter:

$$p = 2\pi R_s$$

By varying it, we obtain a family of 2-spheres that are parallel, one-parameter closed surfaces $\Sigma_2(\rho)$ having a minimum area $A = 4\pi R_s^2$.

The closed surface of minimal area $\Sigma_2(0)$ acts like a throat surface.[4]

But the term g_{tt} of the metric is zero when $\rho = 0$, which leads to the nullity of the determinant of the metric at this point, and the impossibility of defining a system of Gaussian coordinates along this closed surface Σ_2, in other words it is impossible to orientate time and space.

The geometric object emerging from the Schwarzschild solution is therefore not a manifold in the classical sense of the term, but an orbifold containing a singular region $\Sigma_2(0)$ where the object is locally non-orientable. This is not a sphere, but a projective P_2. It was already difficult to create the mental image of two three-dimensional spaces connected by a throat sphere. One imagines the mental efforts that must be deployed if this throat surface becomes Boy's surface…

Such a structure then fits with the Janus model. Indeed, time reversal according to the dynamical group theory, [18] is actually the inversion of energy, hence the inversion of mass.[5] Therefore, the object represents some kind of space bridge connecting two PT-symmetric Minkowski spaces.

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[4] Description of this surface as a throat 2-sphere in [11].

[5] See ref. [18] Part III. *Mechanics* – Chapter 14: *A mechanist description of elementary particles* – Section *Inversions of space and time*, pages 189–193.

About the time of free fall

Classically, events occurring near a black hole are theoretically seen as "frozen in time" for a distant observer. Indeed, when the time of free fall is calculated with the variable *t*, which is supposed to be the time experienced by the distant observer, and is compared to the proper time *s* of the test particle falling onto the black hole, which is the only one intrinsically related to the geometric object, one finds the this: [9]

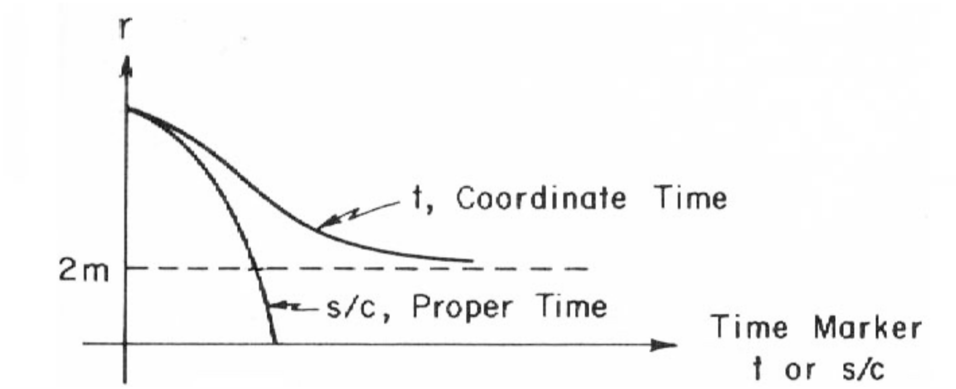

Fig. 17 – Fall toward the origin of a Schwarzschild geometry
in terms of coordinate time *t* and proper time on the test particle *s*/*c*.

This choice of time marker *t* seems necessary. Yet we must bear in mind that the choice of variables with respect to the description of the hypersurface-solution, where the only intrinsic quantity is *s*, remains an *arbitrary choice* representing the physical interpretation of the solution.

We then refer to the Kerr metric according to the formulation given by Boyer and Lindquist. [19] We will replace their space variable ρ with the same Greek letter P in uppercase, to avoid any confusion with the other change of variable used above.

Here, $\mathrm{P} = \sqrt{x^2 + y^2}$:

$$ds^2 = \left(1 - \frac{R_s \mathrm{P}}{\mathrm{P}^2 + a^2 \cos^2\theta} \right) dt^2 - \frac{\mathrm{P}^2 + a^2 \cos^2\theta}{\mathrm{P}^2 + a^2 - R_s \mathrm{P}} d\mathrm{P}^2$$

$$- (\mathrm{P}^2 + a^2 \cos^2\theta) d\theta^2 - \left[(\mathrm{P}^2 + a^2)\sin^2\theta + \frac{R_s \mathrm{P} a^2 \sin^4\theta}{\mathrm{P}^2 + a^2 \cos^2\theta} \right] d\varphi^2 - \frac{2 R_s \mathrm{P} a \sin^2\theta}{\mathrm{P}^2 + a^2 \cos^2\theta} dt\, d\varphi$$

This expression can be particularized, taking $\theta = \pi / 2$. Therefore:

$$ds^2 = \left(1 - \frac{R_s}{\mathrm{P}} \right) dt^2 - \frac{\mathrm{P}^2}{\mathrm{P}^2 + a^2 - R_s \mathrm{P}} d\mathrm{P}^2 - \left[\mathrm{P}^2 + a^2 + \frac{R_s\, a^2}{\mathrm{P}} \right] d\varphi^2 - \frac{2 R_s\, a}{\mathrm{P}} dt\, d\varphi$$

What differentiates this writing of the Kerr metric solution from the classical expression of the Schwarzschild metric? The presence of a cross term $d\varphi\, dt$.

Photons follow null geodesics. Considering azimuthal geodesics at $\mathrm{P} = Cst$, two different values of the speed of light are obtained. Similarly, if we calculate the period of rotation of test particles along circular geodesics, we will get two different values, depending on whether we travel these geodesics clockwise or counterclockwise.

This is classically interpreted as rotational frame-dragging or Lense-Thirring effect. In [9] we read:[6]

> *Loosely speaking, we may think of the rotating source as "dragging" space around with it; in a Machian sense the source "competes" with Lorentzian boundary conditions at the infinity in the establishment of a local inertial frame.*

This situation is inherently related to the Kerr solution. We can wonder, always in the sense of Ernst Mach, if this strong inertial dragging of the reference frame within the ergosphere may represent a phenomenon automatically linked to such extreme situations, so this has to be necessarily taken into account.

But how to introduce a *radial* frame-dragging? Thanks to the change of time variable suggested by Sir Arthur Eddington in 1924: [20]

$$t = t' - R_s\, Log \left| \frac{R}{R_s} - 1 \right|$$

Such a change of temporal variable can be applied either to the Schwarzschild metric or to the Kerr metric. Let's choose here to make this change in the Schwarzschild metric, written in the coordinate system $(t , R , \theta , \varphi)$.

---

[6] Ref. [9] Chapitre 7 – *The Kerr Solution*, page 258.

Therefore:

$$ds^2 = (1 - \frac{R_s}{R})\,dt'^2 - (1 + \frac{R_s}{R})\,dR^2 - 2\,\frac{R_s}{R}\,dt'\,dR - R^2\,(\,d\theta^2 + \sin^2\theta\,d\varphi^2\,)$$

Trajectories of test particles along timelike geodesics can be calculated:

$$d\varphi = \pm \frac{dR}{R^2\sqrt{\dfrac{\lambda^2 - 1}{h^2} + \dfrac{R_s}{h^2 R} - \dfrac{1}{R^2} + \dfrac{R_s}{R^3}}}$$

But the calculation of the free fall time provides a different result depending on whether it is a centripetal radial trajectory $(\,\nu = +1\,)$ or a centrifugal one $(\,\nu = -1\,)$. The radial trajectories correspond to *h* = 0. Then:

$$dt = \frac{\lambda\,R + \,\nu\,R_s\sqrt{\lambda^2 - 1 + \dfrac{R_s}{R}}}{\nu\,(\,R - R_s\,)\sqrt{\lambda^2 - 1 + \dfrac{R_s}{R}}}\,dR$$

When the parameter λ is equal to unity, this corresponds to a particle with zero velocity at infinity. We will place ourselves in a situation close to these conditions. Let's consider a radial trajectory in a situation corresponding to the vicinity of the surface $R = R_s$:

$$dt \simeq \nu\,\frac{r + \nu\,R_s}{(\,r - R_s\,)}\,dr$$

- In a plunging trajectory $(\,\nu = -1\,)$ the free fall time is finite.
- In an escape trajectory $(\,\nu = 1\,)$ it becomes infinite.

We have a *one-way membrane*.

When the test particle crosses the throat surface, time reverses. If we still consider this motion, as it would be perceived by a distant observer made of positive mass,[7] the cross term is now opposite and the metric is written:

$$ds^2 = (1 - \frac{R_s}{R}) dt'^2 - (1 + \frac{R_s}{R}) dR^2 + 2\,\frac{R_s}{R}\, dt'\, dR - R^2 (d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Opposite situation too for radial trajectories:

$$dt = \frac{\lambda\, R - \nu\, R_s \sqrt{\lambda^2 - 1 + \frac{R_s}{R}}}{\nu\, (R - R_s) \sqrt{\lambda^2 - 1 + \frac{R_s}{R}}}\, dR$$

Conversely this time, escape trajectories take place over a finite time, while plunging trajectories are associated with an infinite time.

The passage still behaves as a one-way membrane.

It should be noticed that the opposite sign of the cross term in the metric leads to an ephemeral white hole model in the negative sector, i.e. there, the throat surface still behaves like a one-way membrane, but in the other direction.

Crossing this ephemeral throat surface is therefore a one-way trip. Yet, the flow direction depends on the sector (positive or negative) from which this "hyperspace bridge" is generated.

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[7] In the Janus model, it would not be possible for the distant observer, made of positive mass, to still *see* the motion of these particles after their mass inversion, since negative mass emits negative energy photons that follow null geodesics of the metric $g_{\mu\nu}^{(-)}$. The inverted mass thus seems to disappear from the observer's reference frame, from its own point of view. The throat surface also acts in the Janus model as an event horizon, although a very short-lived one. See references [5] [6] [7].

Conclusion

This black hole chimera is a house of cards and it tumbled down. The "freeze-frame" is a consequence of an arbitrary choice of the temporal variable while at the same time neglecting the frame-dragging phenomenon. In his opening lecture, at the Karl Schwarzschild Meeting in Frankfurt (the city where he was born) in July 2017,[8] Maldacena said:

— *The Schwarzschild solution has confused us over a hundred years and it has forced us to sharpen our views on space and time. It has lead to sharper understanding of Einstein's theory. Experimentally, it is explaining several astrophysical observations. Its quantum aspects have been a source of theoretical paradoxes that are forcing us to understand better the relation between spacetime, geometry and quantum mechanics.*

On the contrary, I believe that this alleged deepening of the conception of spacetime has led cosmologists to depart from the original vision of men like Einstein and Schwarzschild, who had a very profound intuition of geometry and physical phenomena. By choosing to extend spacetime to a phantasmal element, they built an object they called a black hole, endowed with a strange "interior" in which, in the words of specialists "time and space are interchanged one another".

Formerly, the late French cosmologist Jean Heidmann used to say:

Jadis le cosmologiste français feu Jean Heidmann avait coutume de dire :

— *When talking about black holes, common sense has to be left in the cloakroom.*

Such an attitude has led scientists to dissert for half a century on properties of a central singularity that exists only in their imagination, in the mathematical definition of the term.

The theoretical phantasmagoria that preceded the elaboration of such theories ensues from the lack of in-depth reading of Schwarzschild's second paper [2] published in February 1916 (admittedly untranslated for 83 years!) where everything was already put in place, summed up in the Fig. 7 that we reproduce again below.

[8] 3rd Karl Schwarzschild Meeting on Gravitational Physics and the Gauge/Gravity Correspondence (KSM 2017), 24–28 July 2017, FIAS, Frankfurt am Main, Germany.

durch Ausstrahlung) erfolgt.

4. Die Lichtgeschwindigkeit in unserer Kugel wird:

$$v = \frac{2}{3\cos\chi_a - \cos\chi}, \tag{44}$$

sie wächst also vom Betrag $\frac{1}{\cos\chi_a}$ an der Oberfläche bis zum Betrag $\frac{2}{3\cos\chi_a - 1}$ im Mittelpunkt. Die Druckgröße $\rho_0 + p$ wächst nach (10) und (30) proportional der Lichtgeschwindigkeit.

Im Kugelmittelpunkt ($\chi = 0$) werden Lichtgeschwindigkeit und Druck unendlich, sobald $\cos\chi_a = 1/3$, die Fallgeschwindigkeit gleich $\sqrt{8/9}$ der (natürlich gemessenen) Lichtgeschwindigkeit geworden ist. Es

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ist damit eine Grenze der Konzentration gegeben, über die hinaus eine Kugel inkompressibler Flüssigkeit nicht existieren kann. Wollte man unsere Gleichungen auf Werte $\cos\chi_a < 1/3$ anwenden, so erhielte man bereits außerhalb des Kugelmittelpunktes Unstetigkeiten. Man kann jedoch für größeres χ_a Lösungen des Problems finden, welche wenig-

Fig. 7

That is to say:

> At the center of the sphere ($\chi = 0$) velocity of light and pressure become infinite when $\cos\chi_a = 1/3$, and the fall velocity becomes 8/9 of the (naturally measured) velocity of light. Hence there is a limit to the concentration, above which a sphere of incompressible fluid can not exist. If one would apply our equations to values $\cos\chi a < 1/3$, one would get discontinuities already outside the center of the sphere.

Nature has more common sense than one imagines, and arranges to give signs to theorists. Provided they hear them…

The black hole has become the *deus ex machina* of modern times. Computer-generated images are everywhere. We hear, chanted, the sentence:

— *Although there is no observational confirmation, no scientist any longer doubts their existence.*

Which is a complete nonsense from the point of view of the scientific method.

It is overused to fit every occasion. Where does the energy of quasars come from? Black holes, theorists say. Using which mechanism? Nobody knows.

As for quantum gravity, it remains an unachieved discipline, insofar as gravitation has never been quantified. Discoursing on its ins and outs is tantamount to speculating on the outcome of a marriage that has never been consummated.

Theoretical astrophysics must produce models that are likely to be confronted with observations, to be falsified in the sense of Karl Popper. The model we propose leads to the conclusion that the mass of neutron stars is automatically limited below the critical value of 2.5 solar masses, through a mechanism removing any excess of matter via partial mass inversion. The lack of any observation of X-ray emitting objects with a higher mass also militates in favor of this model. The fusion of two subcritical neutron stars could finally explain the observation of gravitational waves, the energy involved corresponding to the mass inversion of the matter in excess, of up to several solar masses.

We predict that stellar black holes will never be observed, simply because they exist only in the imagination of their creators.

We also conjecture that the energy source of quasars in active galaxy nuclei does not rely on black holes and that "supermassive black holes" located in the center of galaxies are very massive relics of such quasars. Work in progress on this subject, that will be presented at a later stage.

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