Complex symplectic mechanics : a toy model for life ?

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Abstract : We show that the classical technique of the symplectic mechanics can be extended to a complex Hermitean space, giving a complex momentum map with complex geometrical attributes : energy, impulsion, spin.

In 1970 the mathematician J.M.Souriau introduces symplectic mechanics [1]. Starting from the isometry group of Minkowski space, i.e. the Poincaré group :

(1) $$\begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix}$$

where L is the Lorentz group, and C the boost

(2) $$C = \begin{pmatrix} \Delta x^\circ \\ \Delta x^1 \\ \Delta x^2 \\ \Delta x^3 \end{pmatrix}$$

he computes the coadjoint action of the group on the dual of its Lie Algebra, which gives the following momentum map :

(3) $$M' = L\,M\,{}^tL + C\,{}^tP\,{}^tL - L\,P\,{}^tC$$

(4) $$P' = L\,P$$

where

(5) $$P = \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

So that the Energy E , the impulsion p and the spin (enclosed in the M matrix) appear like pure geometrical objets, components of the momentum.

Surprizingly this technique can be easily extended to the complex world [2].

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If we replace the Minkowski coordinates $\left\{ x^\circ, x^1, x^2, x^3 \right\}$ by complex coordinates we may form the 4D Hermitean space, whose metric is :

(6) $$ds^2 = dx^\circ * dx^\circ - dx^1 * dx^1 - dx^2 * dx^2 - dx^3 * dx^3$$

This metric is defined on a Hermitean manifold.

Let's now consider the real matrix G :

(7) $$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and the complex Lorentz group defined as :

(8) $$*L\, G\, L = G$$

*L stands for the adjoint of L.

One can show easily that the complex Poincaré group

(9) $$\begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix}$$

where C is th complex boost, is an isometry group of the Hermitean space and can be considered as a complex dynamic group. Surprizingly all classical (matrix) calculations can be extended to such complex framework by simply substituting the adjoint matrices *A to the transpose matrices ${}^tA$. See the annex.

As a result, the complex momentum obeys the laws :

(10) $$M' = L\, M * L + C * P * L - L\, P * C$$

(11) $$P' = L\, P$$

where

(12) $$P = \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

E is a complex energy and p a complex impulsion. The spin, in complex matrix M, is complex too.

The symplectic approach creates a link between pure mathematics and physics. In the case of such complex extension, what could be its physical meaning ?

The Minkowski spacetime can be considered as the real part of a Hermitean space. In Minkowski space time we can consider the null geodesics as the paths of photons, and the non-zero geodesics as the paths of particles with spin.

Notice that in classical physics people only consider the restricted Poincaré group, built on the restricted Lorentz group. In effect the Lorentz group, as defined axiomatically by

(13) $$ {}^{t}L\,G\,L = L $$

owns four connex components.

- L_n is the neutral component, whose elements keep the orientation of time and space.

- L_s elements invert space, not time.

- L_t elements invert time, not space

- L_{st} elements invert both time and space

The first two set form the orthochron subgroup $\{L_o\}$

The next two form the antichron set $\{L_a\}$

The restricted Poincaré group is :

(14) $$ \begin{pmatrix} L_o & C \\ 0 & 1 \end{pmatrix} $$

The corresponding action on the moment is :

(15) $$ M' = L_o\, M\, {}^{t}L_o + C\, {}^{t}P\, {}^{t}L_o - L_o\, P\, {}^{t}C $$

(16) $$ P' = L_o\, P $$

The complete Poincaré group can be written as :

(17) $$ \begin{pmatrix} \lambda L_o & C \\ 0 & 1 \end{pmatrix} \quad \text{with } \lambda = \pm 1 $$

which gives :

(18) $$ M' = L_o\, M\, {}^{t}L_o + \lambda C\, {}^{t}P\, {}^{t}L_o - \lambda L_o\, P\, {}^{t}C $$

(19) $$ P' = \lambda L_o\, P $$

From (19) we see that the time inversion goes with the inversion of the energy , and mass.

The Janus group

A first extension of the restricted Poincaré groupe is :

(20)
$$\begin{pmatrix} \mu & 0 & \phi \\ 0 & L_o & C \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad \mu = \pm 1$$

and acts on the 5D Kalusa space :

(21)
$$\begin{pmatrix} \zeta \\ x^{\circ} \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

with a new boost :

(22)
$$\begin{pmatrix} \phi \\ C \end{pmatrix}$$

Applying the calculation of the action of the momentum, this gives an additional equation :

(23)
$$q' = \mu q$$

q is the additional physical quantity corresponding to the translation along the fifth dimension . It is identified to the electric charge. The inversion of the charge goes with the inversion of the fith dimension [3] .

Now we can shift to the Janus group :

(24)
$$\begin{pmatrix} \lambda\mu & 0 & \phi \\ 0 & \lambda L_o & C \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad \mu = \pm 1 \quad \text{and} \quad \lambda = \pm 1$$

which acts on the janus bimetric space ([7] , [14])

After Janus model the universe is a manifold with two metrics $g_{\mu\nu}^{(+)}$ and $g_{\mu\nu}^{(-)}$ which refer to positive and negative mass. After the Janus coupled field equations (&&&) :

(25)

$$R^{(+)}_{\mu\nu} - \frac{1}{2}R^{(+)}g^{(+)}_{\mu\nu} = \chi\left[\ T^{(+)}_{\mu\nu} + \varphi\sqrt{\frac{g^{(-)}}{g^{(+)}}}\ T^{(-)}_{\mu\nu}\ \right]$$

$$R^{(-)}_{\mu\nu} - \frac{1}{2}R^{(-)}g^{(-)}_{\mu\nu} = -\chi\left[\ \phi\sqrt{\frac{g^{(-)}}{g^{(+)}}}T^{(+)}_{\mu\nu} + T^{(-)}_{\mu\nu}\ \right]$$

Newtonian approximation gives the gravitational interaction laws :

- Masses with same sign mutually attract through « Newton's law »
- Masses with opposite signs mutuelly repel through « anti-Newton »

A toy model for life.

We may consider the complete complex Poincaré group :

(26) $$\begin{pmatrix} \lambda L_o & C \\ 0 & 1 \end{pmatrix} \quad \text{with } \lambda = \pm 1$$

where L_o and C are complex and λ real.

Concentrate on the movement of masses and photons in the real part of the Hermitean space, i.e. the Minkowski space. They corresponds to physical matter, moving in the « real physical world ».

Now, consider the imaginary part of our Hermitean space. It becomes a « metaphysical space », inhabited by imaginary masses (positive and negative). .

The real world is inhabited by real masses, positive and negative. They mutually repel. Gravitational interaction makes them form positive clusters (galaxies, clusters of galaxies, stars, planets). Nuclear interaction gives a wide set of atoms. Electromagnetic forces produce molecules and biomolecules.

What could we find in our metaphysical world ? We may speculate that some interaction forces act, among the different species. Let's assume that positive imaginary masses + im and negative imaginary masses –im obey a set of laws similar to the one of the real matter. The will tend to form clusters.

Let's continue to play with such toy model. We decide that the positive species would represent « the will » and negative species the « evil ». We could also imagine to introduce « imaginary carriers », similar to photons. A very rich and complex mixture.

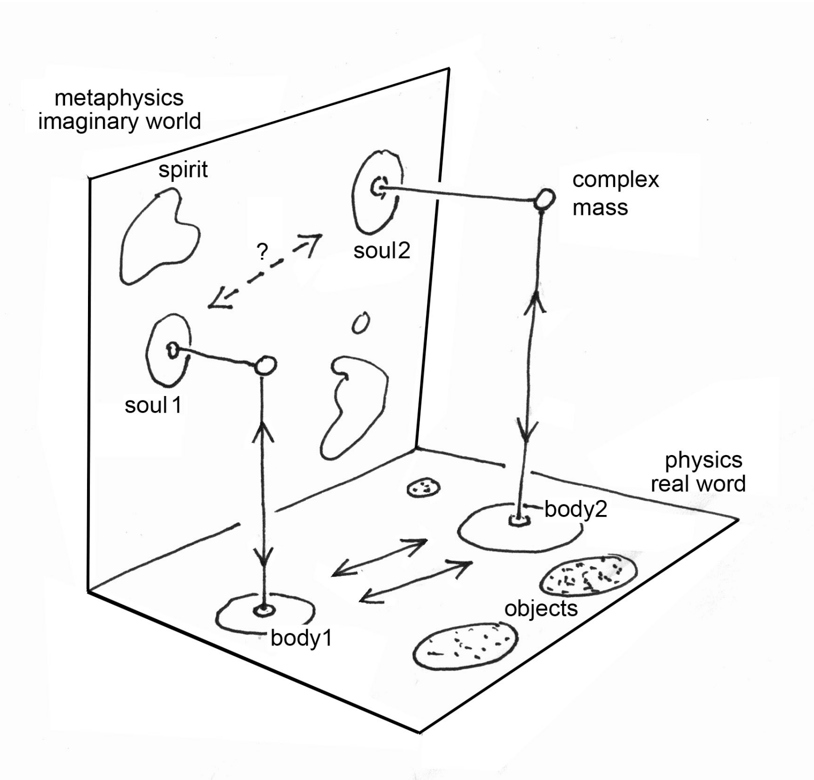
In the real world positive and negative particles don't glue together. The stand far apart. This corresponds to the very large structure of the universe when, after the discoupling, negative mass clusters form, confining positive mass in the remnant place, which form a lacunar structure, tu be compared to joint soap bubbles ([8], [11]).

The fate of the real world is ruled by physical laws, corresponding to a limited set of equations.

We could imagine that the metaphysical world's fate could be ruled by unknown metaphysical rules, that would produce different kinds of structures. We could compar some of them to « souls » or « spirits ».

Now, what about life ?

In this toy model a living creature would be a structure composed by a body, i.e. un certain set of linked positive masses , plus a soul. If the model is based on a 5D complex space, we may include some metaphysical electromagnetic forces like interactions that would help the formation of complex objects, composed by the two kinds of masses, positive imaginary and negative imaginary, i.e. made of « will » and « evil ».



On the figure we have figured two living creatures as associations of (body + soul). Then the « death » correponds to the breaking of the link. Conversely the birth would correspond to the making of the link.

It's difficult to modelize conscienceness based on real matter. It could be possible using the imaginary particles and fields, and metaphysical laws and interaction forces.

We let the reader to build his own toy model. A job for philosophers.

References :

[1] J.M.Souriau : Structure des systèmes dynamiques. Dunod Ed. France, 1970 and Structure of Dynamical Systems. Boston, Birkhaüser Ed. 1997

[2] J.P.Petit : A Symplectic Cosmological Model. Progress in Physics, Vol. 14, jan. 2018
http://www.ptep-online.com/2018/PP-52-09.PDF

[3] J.M.Souriau : Géométrie et Relativité (in french). Ed. Hermann 1954

[4] J.P.Petit, G.D'Agostini : Cosmological Bimetric model with interacting positive and negative masses and two different speeds of light, in agreement with the observed acceleration of the Universe. Modern Physics Letters
A, Vol.29 ; N° 34, 2014 ; Nov 10th DOI :10.1142/So21773231450182X

[5] J.P.Petit & G.D'Agostini : Cancellation of the singularity of the Schwarzschild solution with natural mass inversion process. Mod. Phys. Lett. A vol. 30 n°9 2015Doi:10.1142/S0217732315500510

[6] J.P.Petit & G.D'Agostini : Lagrangian derivation of the two coupled field equations in the Janus Cosmological Model. Astrophysics and Space Science 2015, 357 :67Doi : 10.1007/s10509-015-2250-6

[7] J.P.Petit, G.D'Agostini : Negative Mass hypothesis in cosmology and the nature of dark energy. Astrophysics And Space Science,. *A* **29**, 145-182 (2014)

[8] J.P.PETIT, Twin Universe Cosmology, Astrophys. and Sp. Science, **226**, 273-307, 1995

[9] J. P. Petit, Univers Enantiomorphes à flèches du temps opposées, CRAS du 8 mai 1977.

t. 285, pp. 1217–1221 (1977).

[10] J. P. Petit, Univers en interaction avec leur image dans le miroir du temps, CRAS du6 juin, t. 284, série A, pp. 1413–1416 (1977).

[11] J.P.Petit, G. D'Agostini, N.Debergh : Physical and mathematical consistency of the Janus Cosmological Model (JCM). Progress in Physics 2019 Vol.15 issue 1

[12] J. P. Petit, P. Midy and F. Landhseat, Twin matter against dark matter, int. Conf. onAstrophysics and Cosmology, Where is the Matter?, Tracing Bright and Dark Matter with the New Generation of Large-Scale Surveys (Marseille, France, June 2001).

[13] J. P. Petit, Nuovo Cimento B. The missing mass problem. **109**, 697 (1994).

[14] J. P. Petit and G. D'Agostini, J.P.Petit, Negative mass hypothesis in cosmology and the nature of dark energy. Astrophysics and Space Science, 2014 sept. 20 th

[15] J.P.Petit, P.Midy and F.Landsheat: Twin matter against dark matter. Intern. Meet. on Atrophys. and Cosm. "Where is the matter ? ", Marseille 2001 june 25-29

Annex

Consider the complex Poincaré group

(1) $$g = \begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix}$$

The element of its complex Lie algebra is

(2) $$Z = \begin{pmatrix} G\omega & \gamma \\ 0 & 0 \end{pmatrix} \quad \omega = \text{antisymmetric matrix}$$

We form

(3) $$Z' = g^{-1} Z \; g$$

we get

(4) $$\begin{pmatrix} G\omega' & \gamma' \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} L^{-1} G\omega L & G\omega C + L^{-1}\gamma \\ 0 & 0 \end{pmatrix}$$

as $G\,G = I$ and $L^{-1} = G\,{*}L\;G$

(5) $$\omega' = {*}L\omega L$$

(6) $$\gamma' = G\;{*}L\omega C + G\;{*}L G\gamma$$

The complex momentum of the complex Poincaré group will be denoted :

(7) $$\mu = \{\; M\,,\, P \;\} \text{ with } {*}M = -M$$

and will be defined by the identity :

(8) $$\mu \equiv \frac{1}{2}\,\mathrm{Tr}\big(M\omega\big) + {*}P G\gamma$$

Introduce the duality relationship :

(9) $$\frac{1}{2}\,\mathrm{Tr}\big(M\omega\big) + {*}P G\gamma = \frac{1}{2}\,\mathrm{Tr}\big(M'\omega'\big) + {*}P'\,G\;\gamma'$$

We get

(10) $$\frac{1}{2}\,\mathrm{Tr}\big(M\omega\big) + {*}P G\gamma = \frac{1}{2}\,\mathrm{Tr}\big(M'\,{*}L\omega L\big) + {*}P'\,L\omega C + {*}P'\,{*}L G\gamma$$

The identification on the γ terms gives

(11) $$*P = *P'*L \quad \rightarrow \quad P = L\,P'$$

In the trace one can achieve a circular permutation

(12) $$\mathrm{Tr}\left(M'*L\omega L \right) = \mathrm{Tr}\left(L\,M'*L\omega \right)$$

The identification on the ω terms gives :

(13) $$\frac{1}{2}\mathrm{Tr}\left(M\omega \right) = \frac{1}{2}\mathrm{Tr}\left(LM'*L\omega \right) + *P'*L\omega C$$

The second term of the second member is equal to thje product of a line-matrix by a column-matrix. This being equal tu the reversed product. Hereafter, schematically, the product of a line-matrix by a column-matrix :

(14) $$*P'*L\omega C = \mathrm{Tr}\left(*L\omega C*P' \right)$$

In the trace we can achieve a circular permutation :

(15) $$*P'*L\omega C = \mathrm{Tr}\left(C*P'*L\omega \right)$$

whence

(16) $$\frac{1}{2}\mathrm{Tr}\left(M\omega \right) = \frac{1}{2}\mathrm{Tr}\left(L*M*L\,\omega \right) + \frac{1}{2}\mathrm{Tr}\left(C*P'*L\,\omega \right)$$

where is an antisymmetric matrix. We know that a matrix' trace is equal to the product of another matrix by a symmeytrical matrix. Any matrix can be symmetrized or antisymmetrized. In addition thje trace of the product of a matrix by an antisymmetric matrix is zero. Whence :

(17) $$\mathrm{Tr}\left(A\omega \right) = \mathrm{Tr}\left[\mathrm{antisym}\left(A\right) \times \omega \right]$$

We can apply that to the matrix $C*P*L$ why we take the trace of its product by an antisymmetric matrix ω

(18) $$C*P*L = \mathrm{sym}\left(C*P*L \right) + \mathrm{antisym}\left(C*P*L \right)$$

But

(19) $$\mathrm{Tr}\left[\mathrm{sym}\left(C*P*L \right) \times \omega \right] = 0$$

So that :

(20) $$\mathrm{Tr}\left[C*P*L\,\omega \right] = \mathrm{Tr}\left[\mathrm{antisym}\left(C*P*L \right) \times \omega \right]$$

(21) $$\mathrm{antisym}\left(C*P*L \right) = \frac{1}{2}\left[C*P*L + *\left(C*P*L \right) \right]$$

(33) $$*\left(C\ *P\ *L \right) = *\left(P\ *L \right) *C = L\ P\ *C$$

(34) $$\text{antisym}\left(C\ *P'\ *L \right) = \frac{1}{2}\left(C\ *P'\ *L\ - L\,P'\ *C \right)$$

(35) $$\text{Tr}\left(C\ *P\ *L \right) = \frac{1}{2}\,\text{Tr}\left(C\ *P'\ *L - L\ P'\ *C \right)$$

Finally

(37) $$M = L\,M'\,*L + C\ *P'\ *L - L\ \ P'\ *C$$

(38) $$P = L\,P'$$

We can interchange M and M' , P and P' and we get the same forlae for the complex moment map, just changing « transposed » by « adjoint » :

$$\boxed{\begin{array}{c} M' = L\,M\ *L + C\ *P\ *L - L\ \ P\ *C \\ \\ P' = L\,P \end{array}}$$

We can put the momentum into a matrix form :

(39) $$\mu = \begin{pmatrix} M & -P \\ *P & 0 \end{pmatrix}$$

and the action of the complex Poincaré group becomes

(40) $$\boxed{\mu' = g\,\mu\,*g}$$